CONCEPTS AND CONTEXTS IN LEARNING MATHEMATICS

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Modern primary mathematics curricula explicitly aim to develop children's understanding of basic mathematical concepts, in contrast to the more traditional rote learning of procedures. However there has been continuing concern that many children still do not achieve an acceptable level of understanding of these mathematical concepts and most seem to have difficulty generalising their understanding to relevant contexts.

An investigation of the mathematical understanding of twelve students during their transition from primary to secondary school suggests that there may be little educational value in basing curricula on a presumed common learning hierarchy of concepts within mathematics which can be transferred to the learners' minds through a controlled sequence of learning experiences. The conceptual understanding that students were able to demonstrate (though quite limited) revealed that their mathematical conceptualisations often consisted of unique interpretations associated with personally perceived contexts, rather than common and clearly recognisable generalisations.

For many years the development and modification of primary mathematics curricula and teaching methodologies have been explicitly aimed at facilitating children's understanding of basic mathematical concepts (Ellerton & Clements 1988). In Britain, the Cockcroft Report (Cockcroft 1982), which has also had wide influence in Australia, particularly drew attention to the need for a detailed and careful approach to teaching mathematics in the early years "in order that children may develop confidence and understanding" (p83).

In much of the related documentation, the term concept is used in such a way as to imply that there are identifiable entities within mathematics itself which can be transferred to the learner's mind in a fixed learning sequence appropriate to various stages of development. If properly taught and understood these concepts then are presumed to become accessible to the learner as part of a total mental mathematical construct, the characteristics of which should be similar among successful learners of mathematics.

> The child's level of understanding of a concept depends upon the quality, extent and connectedness of verbal (written and oral), concrete, pictorial and symbolic forms of representation.(Western Australian Ministry of Education 1989 p23)

Clearly the intention has been to emphasise the teaching of mathematics in a meaningful way with the child actively making sense of the ideas presented in contrast to the more traditional approach where rote learning of procedures constituted a major focus of classroom practice. However, the frustration for many teachers who seriously attempt to teach conceptual understanding is that children often appear to have great difficulty, not only in developing an understanding of these concepts initially, but in maintaining their understanding during later learning stages and, more particularly, in generalising the concepts to what seem obviously related applications and contexts (Hart 1989; Swan 1990).

The consistency of these difficulties in learning mathematics suggests the need to question the assumption that it is possible to identify a hierarchy of 'concepts' within the discipline which is appropriate to the learning sequence of all individuals. Indeed Ernest (1991) and Sierpienska (1990) argue strongly against this notion and support the proposition that conceptual understanding in mathematics arises out of the personalised and idiosyncratic sense and meaning the learners assign to the contexts of use to which they are exposed, suggesting that a well defined common learning route through sequentially ordered mathematical 'concepts' of increasing difficulty cannot be pre-determined. Kameenui and Griffin (1989) also question the educational appropriateness of focusing on mathematical 'concepts' rather than the unique contextual associations they believe each individual student intuitively develops during early learning experiences.

Clarke's (1989) findings provide some research support for these ideas. His intensive study of the mathematical behaviour of ten students during their transition from primary to secondary school revealed that the mathematical behaviour of the students was highly idiosyncratic and clearly involved complex interactions between students and their perceptions of the learning environment as well as important social factors. All students were in the same classroom during Year Seven and Year Eight and therefore were exposed to the same mathematical experiences at school, and yet each had developed a unique perspective and exhibited widely differing characteristics in their approach to understanding mathematics. Clarke noted that any explanation of these differences must accommodate "the essential individuality of the learning process" (Clarke 1989 p335).

This paper further explores the individuality of students' mathematical conceptualisations with a particular focus on students' ability to contextualise the basic number 'concepts' introduced early in primary school and its relationship to understanding later more abstract ideas. Interview data obtained from students at the end of their primary schooling have been examined and aspects of the students' conceptualisations of number operations are reported and discussed. The implications for primary curricula design are also considered.

SOURCE OF THE DATA

The data reported in this paper were obtained during interviews carried out as part of an investigation into the mathematical understanding and attitudes of students during their transition from primary to secondary schooling (Tomazos 1991). At the end of 1990 all 216 Year Seven students at nine metropolitan primary schools in Western Australia were surveyed using a written questionnaire to determine their attitudes to the main subject areas. From the data obtained, a number of students were identified as having exhibited very strong feelings, both negative and positive, specifically towards mathematics. Of these students six girls and six boys were chosen, with three of each gender showing very positive responses and three of each gender showing very negative responses to mathematics.

The twelve students were interviewed for approximately forty minutes on two occasions, once at the end of Year Seven primary school and once midway through Year Eight, their first year of secondary schooling. Students were questioned about their perceptions of and attitudes towards mathematics and their responses were sought (using a clinical interview approach) to a range of mathematical items designed to examine the students' competence with and understanding of fundamental number operations and relationships. The purpose of the original study was to investigate possible relationships among students' gender, their perceptions and understanding of mathematics,and their affective responses to it.

Because the research focus had been on finding similarities in mathematical behaviours, much interesting data which clearly supported Clarke's (1989) findings that students' mathematical behaviour was highly idiosyncratic had to be set aside at the time. It is largely these data which are now reported and discussed in this paper.

REPORT AND DISCUSSION

Children are generally introduced to the 'basic number concepts' during the first years of schooling through whole number counting activities and the simple combining and separating of sets, extending to multiplication (usually as repeated addition), then division (often in the context of sharing). At some point in these first few years children are exposed to the symbolic representations of these concepts and are usually asked to practise computations using the symbols alone, initially with one digit whole numbers, then (in line with the hierarchical structure of the syllabus) with increasingly larger and more complex numbers at each year level, as new 'concepts' (eg 'place value', 'decimals' etc) are introduced.

Almost all of the items presented in the interviews required application of various number concepts specified in the Western Australian primary mathematics syllabus. Some of the items were presented to the students as decontextualised symbolic representations of operations and students were asked to mentally calculate the items if possible, explain their processes and describe an appropriate context. Ten items included large or more complex numbers (see Appendix A) but would be quite easy to calculate using simple informal mental strategies based on conceptual and contextual understanding.

However, most students actually relied solely on using the standard written algorithmic procedures as their preferred mental method, which involved a very complicated mental process of visualising the numbers, re-arranging them vertically and then mentally performing the steps involved. While only 24% of the attempted items were completed using mental strategies based on conceptual knowledge, the success rate for items completed in this way (68%) was consistently higher than for items in which a complex mental 'written' procedure was attempted (success rate 55%).

The difficulty here is that when children do manipulate numbers successfully using the standard procedures, it is impossible to make valid inferences about their conceptualisations of the numbers or the operations involved, although assumptions are often made in the classroom. It would be very easy to assume, for example, that if students can 'do' *15.01 + 14.99* successfully, they must have a conceptual understanding of decimal numbers and that their conceptualisations of these are similar to all other successful students. Certainly this assumption could not be supported by the students in this study. While the overall success rate for this item was 80%, none of the students were adequately able to explain the meaning of *0.5* or the difference between *0.09* and *0.1* (presented elsewhere in the interviews) even though by Year Eight most students could say that the former was the same as a half and *0.1* was larger than *0.09*. The actual explanations offered revealed a wide range of conceptualisations and personalised perceptions of decimal fractions, none of which could be interpreted as a complete, or even partially complete mental concept for decimals, and differing sufficiently from one another to add support to Ernest's (1991) beliefs about the way in which conceptual understanding develops in mathematics.

Ernest (1991 p241) defines the term *concept* in two ways, firstly in its narrow sense as a simple unitary mental object which can be thought of as a single item or idea, which would only involve simple acts of discrimination. For instance understanding a concept for 'water' in this sense would require the recognition of objects which are "water" and objects which are not. A second broader use of the term 'concept' involves a much more complex mental structure which consists of a number of the simpler concepts as well as the relationships between them. A concept for 'water' in this broader sense might involve a wide range of mental connections and interrelationships with related conceptual structures associated with 'rain', 'ice', 'life', 'chemistry', and 'fire', to name but a few, and would arise out of the personalised contexts, experiences and purposes associated with water up to that point in a person's life. While many of the mental constructs for 'water' may well be shared with others, it is difficult to conceive of a person learning or understanding this complex personalised conceptualisation of water as a complete mental entity of meaning, developed by means of a conceptual pathway common to all learners.

And yet, while most mathematical 'concepts' discussed in the educational literature associated with curricula materials clearly appear to fall within this second broader definition of the term, it is assumed in these curricula that it is not only possible to identify within mathematics suitable discrete conceptual 'chunks' which can be conveyed in their entirety to the learner, but that there is a recognised sequential itinerary of learning stages appropriate for all students into which these conceptual 'chunks' may be slotted. In contrast to this Ernest (1991) claims that "a learner's use of a concept must necessarily be within some context, so the concept is linked to its contexts of use" (p240) and that the learner's "grasp of a concept grows according to the range of contexts of use that are mastered"(p241) during a learning process in which the learner idiosyncratically perceives and attends to personally chosen significant events and relationships. Thus the idea of two learners understanding the same concept can only be contemplated in terms of the similarities of their behaviour in response to a particular mathematical situation or context. Because of the highly personalised nature of their conceptual structures, another seemingly similar mathematical context may well elicit entirely different and quite divergent behaviour from the two learners.

Certainly the students interviewed demonstrated that context could be a powerful determinant of success when standard computational procedures were forgotten. Penny (12:3), for example, had previously mentally calculated *1501 + 1499* using a standard procedure but could see no way of dealing with *15.01 + 14.99*, when the item was presented soon after.

P: Oh, no... I hate decimals.
I: Is there an easy way you can think about them?
P: No (groan) I can't do decimals, would it be...? (pause, shaking head)

Her response changed when she was asked if it would be easier to handle if she thought of the item as money; fifteen dollars and one cent add fourteen dollars, ninety nine.

P: It is, I suppose so... [pause] Yeah!....
I: Okay, well how would you go about that?
P: I'd take the one cent and add it to that to make it one dollar, and then add fifteen and fourteen and one, which is thirty dollars.

The importance of an appropriate context for making the mathematics accessible suggested that students' conceptualisations of the basic operations could be usefully examined by looking at the contexts students provided for some of the simple whole number items included in the interview shedules. These consisted of all combinations of the numerals 'five' and 'ten' using the four operations, (ie *10 + 5, 5 + 10, 10 – 5, 5 – 10, 10* x *5, 5* x *10, 10 ÷5,* and *5 ÷10*). The items were presented one at a time in the above order and students were asked to give an answer, then provide an appropriate 'real life' context for each. .

Of particular interest were the students' responses to two of these items, 5 – 10 and 5 ÷ 10, both of which an hierarchical approach to curriculum development would exclude from the primary syllabus on the basis that they involve difficult and abstract mathematical concepts which could not be understood until later stages when students have acquired the prerequisite concepts. As the students in the study were at the stage when it is assumed they are ready to understand these concepts, it was thought important to investigate how the students' conceptualisation of subtraction and division would be extended to deal with the items, and the role context played in the students' ability to make mathematical sense of the them.

The items *5* x *10* and *5 + 10* were often described by the students as being the "same" as the previous items (*10* x *5* and *10 + 5*) "just the other way around" suggesting students understood commutative properties, but these properties were also assumed for division and subtraction. as demonstrated by responses to the items *5 ÷ 10* and *5 – 10*. Only when encouraged to articulate a context which reflected the "other way around" did some students realise there were inconsistencies. For example.

Nigel (12:7) Year Eight
I: Okay, what about ten take five?

N: Oh, if you have ten smarties and then you give me five, then you've got five and I've got five.
I: What about five take away ten?
N: That's the same, just done around the other way.
I: Is it? So it would work out the same way, would it?
N: Yeah, so if I've got, no he's got, um, I can't really work that one out. Um, no, it's just the same but you can't tell a story about that one.
I: Have you got an answer for it?
N: Yeah, it's five.
I: It's still five?
N: Yes.

This can be related to the learning difficulties which Kameenui and Griffin (1989 p578) suggest arise because students are likely to intuitively associate newly introduced concepts with those previously taught and will often continue to operate on the basis of earlier conceptualisations for as long as possible. A teacher often assumes evidence for continued conceptual growth is demonstrated when students are successfully applying a learned 'concept' within the framework of the controlled contexts which characterise a hierarchical learning sequence of concepts. The student, in fact, may be relying only on those personalised associations and interpretations that are derived from the contexts in which a concept is first introduced and may well be able to continue to operate successfully on that basis throughout a number of steps of presumed increasing difficulty without significantly adding to his or her developing conceptual framework. The problem reveals itself when a new application is introduced which, according to the syllabus, should require a small conceptual step but, for the child, often demands an entirely new framework of contextual associations.

There is some evidence for this in the case of students who could, in fact, accurately articulate a mathematical interpretation of $5 \div 10$, but who were unable to supply an answer nor a suitable context without prompting, even though the relevant fraction concept should be available to them at this stage of their schooling. They seemed, in fact, to be restricted by the contextual associations they had just provided for $10 \div 5$, along with their preferred contextual conceptualisation for division. For example, Penny (12:9) during the Year Eight interview, conceptualised the initial division item as representing a quotition (or grouping) situation. Even with prompting, she seemed unable to move from this conceptualisation to partition (or sharing) which would be needed to make contextual sense of $5 \div 10$.

Penny (12:9) Year Eight

P: ($10 \div 5$) You have ten people and you have to work out how many groups of five you could have.
I: Okay, what about five divided by ten?
P: You can't do that either. (referring to the earlier item $5 - 10$)
I: Not even if I said that was five dollars?
P: And that was ten dollars?
I: Well, was it? I mean would that make sense?
P: Yeah, but I remember last time that you said five dollars and ten cents wouldn't be right because they are different.
I: What about five dollars and ten people?
P: Yeah, I suppose you could really. (pause) I don't know. (shaking head)
I: You're still not comfortable with it, are you?
P: No.

Note that Penny, even when prompted with "five dollars" as a starting point, maintained her initial conceptualisation of division by suggesting groups of ten dollars would be needed. She indicated by her response that she may have even considered the more feasible possibility of separating five dollars into groups of 10 cents, but rejected this because of a remembered mathematical 'restriction', but was not able to re-conceptualise division as a sharing situation. What is particularly interesting in Penny's case is that seven months earlier, during the Year Seven interview, she was able (with a little more prompting) to interpret the *5 ÷ 10* item appropriately.

Penny (12:2) Year Seven

P:	(10 ÷ 5) Yeah, if you've got ten dollars and you want to know, if you, you want to buy something like, Christmas presents, for two dollars each, then you can work out how many presents you gotta buy.
I:	But you know the five first, so is that how many presents that you want to buy?
P:	Yeah, well maybe if you've got ten dollars and you want to buy five presents, and you've got to work out how much money you want to spend on each present.

Note here that the initial context given described 10 ÷ 2 (not *10 ÷ 5*) and was conceptualised as a quotition (grouping) situation. When Penny restated the situation to accurately reflect *10 ÷ 5*, the identical context was used (ie $10 to purchase 5 presents at $2 each) but by choosing to keep each component constant, she was actually forced to describe the situation as partition (sharing), although this seems not to be her preferred choice of a context for division. This re-contextualisation of *10 ÷ 5* did seem to provide a basis upon which she could more easily conceptualise *5 ÷ 10* when prompted with a similar partition situation, even though she initially rejected the possibility that the item could be answered or that a suitable 'story' could be given.

P:	Five divided by ten you can't do!
I:	Do you think your teacher or anyone could do it? (P: No)
I:	What about a story....is there no way to think about that one? (P: No!)
I:	What about if you ask yourself the same question....what about if it was, five dollars and you wanted to buy ten presents, is that possible to think about?
P:	Mm yeah, I think so.
I:	So how much would you spend on each.
P:	Fifty cents.
I:	What fraction of a dollar is fifty cents?
P:	Half.
I:	So do you still think there's no answer, or is it possible.
P:	No, it's possible.

Naomi also provided an example where the student's personal conceptualisation of division and the context that was initially provided determined success with the unfamiliar item. During the Year Seven interview she initially struggled to provide a clear context or show whether she viewed *10 ÷ 5* as reflecting quotition or partition. Even though she clearly knew how to deal with the item numerically, she had difficulty contextualising it and finally saw it as ten dollars divided by five dollars would be two dollars, a conceptualisation of division that was restricted to whole numbers unrelated to a useful context. When encouraged to explain her context, she said a little impatiently, "Oh, its just ten dollars, 10 divided by 5 is 2!" Her response to *5 ÷ 10* reflected this difficulty in conceptualising an appropriate division context and clearly showed that she believed that the dividend must be larger than the divisor. She claimed it was impossible because "ten can't go into it". When asked if her teacher would have an answer, she said no and added that "she'd say it's not a sum, not a real sum".

By Year Eight Naomi had developed a conceptualisation of division which then enabled her to adequately deal with both items. Her context for 10 ÷ 5 was stated as a sharing situation ($10 divided between five people), so that when asked to consider *5 ÷ 10* she repeated the context appropriately adapted, but then was not immediately able to adapt her conceptualisation of division as requiring a larger dividend. The context itself then provided the means for her to re conceptualise her understanding of division.

Naomi (12:7) Year Eight

N:	Um. If you had, like, five dollars and then there are ten people you would have to try and divide that.
I:	Would you be able to?
P:	Afraid not.
I:	Think about it. Think about really doing it.
P:	Oh, that would be point five.

The responses of both Naomi and Penny together with numerous other examples discovered in the interview data, clearly support the notion put forward by Ernest (1991) that conceptual understanding in mathematics develops as an idiosyncratic, contextually situated process in which each student's conceptualisations consist of quite unique mental structures.

Sierpienska's (1990 p27) analysis of the process which may be involved in understanding a mathematical concept is also clearly supported by the data. She postulates that the learner extracts mathematical meaning through successive 'acts of understanding' which seek out relationships between the initial 'sense' conveyed by a situation or communication and its perceived 'references' (those other events, situations, words etc to which the initial event may 'refer'). While the initial sense conveyed by a mathematical situation may be consistent among learners, the acts of understanding a student carries out to deal with it are likely to involve references which are extremely personal and may vary for an item such as *15.01 + 14.99* from the visualisation of a real context through to the memory of a previously encountered symbolic representation perceived of as requiring a similar response.

It is clear, for example, that Penny and Naomi made similar sense out of the items presented in that each recognised the requirement for a division operation and could provide a mathematically correct response for 10 ÷ 5. Likewise the initial sense made of 5 ÷ 10 was similar for both students in that it was considered mathematically impossible to solve. However, the references perceived by each student and the acts of understanding they exposed in response to prompting, seemed to differ quite markedly according to their individual perceptions of division in a real context. It seems clear, therefore, that their conceptualisations of division can not usefully be thought of as resembling a common 'concept'. Each may be able to "do" the operation involved, but the route by which their present understanding was attained and the future direction of their conceptualisations are likely to be quite divergent.

In a classroom situation, the diversity of the references each child uses to move from this initial sense to an acceptable answer may well be invisible to the teacher and are likely only to be uncovered in children whose consistently inappropriate responses invite closer scrutiny. These children are then described as having misconceptions, misunderstandings or being in some way deviant in the development of their conceptual structures, the characteristics of which are believed to be quite different from the assumed conceptual homogeneity of their successful counterparts. Whereas, according to both Ernest and Sierpienska, and supported by the data described in this paper, the development of each child's conceptualisation of mathematical ideas is quite idiosyncratic in nature, as is the mental processes involved in operating mathematically, and these cannot easily be accommodated within the current hierarchical structure of most primary mathematics curricula.

CONCLUSIONS AND IMPLICATIONS

In this paper an attempt has been made to justify the argument that the development of conceptual understanding in mathematics is a very personal and complex mental process that cannot be separated from the learners' perceptions of the contexts in which mathematical ideas are experienced, nor can the character or course of these mathematical conceptualisations be standardised across learners.

The hierarchical structure of a typical primary mathematics curriculum is seen to be inconsistent with this approach in that its design seems centrally focused on the identification of fundamental mathematical 'concepts' and the dissection of these in such a way as to simplify their transfer to the learners' minds in a controlled sequence of learning activities. Contexts of use in such an hierarchical learning structure seem to be very narrowly and superficially incorporated primarily as a means of exemplifying or illuminating concepts for the learner, but also to provide practice in the application of specific concepts after they have been 'understood'. To follow such a curriculum a teacher would likely be pre-occupied with controlling the difficulty level of each mathematical learning experience in order to simplify the sequential acquisition of these concepts.

The responses of the students interviewed support the viewpoint that this curricula organisation actually encourages classroom teachers to provide an impoverished mathematical learning environment. The range and quality of the references and associations available to the students during the development of their personalised conceptualisations of mathematical ideas seem to have been severely restricted, narrowing their mathematical repertoire to specific fragmented contexts of use without access to any of the broader conceptual structures which might have allowed the students to perceive appropriate references and relationships associated with the particular range of mathematical situations presented.

An alternative primary curricula framework could more profitably focus on providing students with opportunities to begin engaging in a wide range of mathematical encounters in real contexts with real purposes at a very early age. The types of contexts provided should be dictated by the interests and life experiences of the students involved, as well as the potential richness of the mathematical ideas which *may* be exposed by the situation. In such a rich environment students can be encouraged to enter into genuine mathematical discourse at a very early age, not only without fear of confusion, but purposefully exposing the ambiguities and complexities inherent in mathematising in real contexts. Students then have access to a fertile mathematical environment in which they are expected to make personal sense of their experiences and gradually approximate adult mathematical behaviour, parallel in fact to the whole language approach to literacy learning which is itself based on children's natural 'problem solving' approach to learning.

Because such a mathematics curriculum would be centrally concerned with real contexts for real purposes, the conceptualisation process the learners engage in would be very similar to that used by mathematicians and experienced problem solvers when approaching new ideas, and the teacher could expect to be the students' collaborator in this process of solving problems. Problem solving in this sense is not conceptualised as an activity which can be separated from other mathematical processes, but rather is inherent in mathematical activity *per se*.

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Appendix A

List of items involving operations and large or complex numbers from the interview schedule.

1.	1499 + 1501	5.	1.49 + 1 1/2	9.	2.5 x 0.2
2.	3000 – 1499	6.	250 x 200	10.	5/8 + 7/8
3.	15.01 + 14.99	7.	25 x 0.2		
4.	15.01 + 1.499	8.	2.5 x 200		